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(Residential Autonomous College under University of Calcutta)

B.A./B.SC. FIRST SEMESTER EXAMINATION, DECEMBER 2011

FIRST YEAR

MATHEMATICS (General)

Date : 19/12/2011

Time : 11am – 2pm

Paper : I

Full Marks : 75

[Use separate answer-books for each group]

Group–A

1. Answer **any five** questions: 5x5 = 25

a) For any positive integer n (>1) show that all the solutions of $z^n = (1+z)^n$ represent points in the Argand plane, which are collinear. 5

b) Express i^i in the form $a+ib$, where a, b are real numbers. Prove that $\sin(\log i^i) = -1$. 3+2

c) Find the values of k for which the equation $x^3 - 9x^2 + 24x + k = 0$ may have multiple roots. Find also the roots of the equation. 2+3

d) (i) Show that the equation $3x^5 - 4x^2 + 8 = 0$ has at least two imaginary roots. 3

(ii) Using synthetic division, find $f(x+1)$ if $f(x) = x^5 + 5x^3 + 3x$. 2

e) If α, β, γ be the roots of $x^3 + qx + r = 0$, form the equation whose roots are $\alpha^2+\beta^2$, $\beta^2+\gamma^2$, $\gamma^2+\alpha^2$. Hence find $(\alpha^2+\beta^2)\cdot(\beta^2+\gamma^2)\cdot(\gamma^2+\alpha^2)$. 4+1

f) Solve the equation $x^3 + 6x^2 - 12x + 32 = 0$, using Cardan's method. 5

g) Define orthogonal matrix. Determine a, b, c when $A = \begin{pmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{pmatrix}$ is orthogonal. 1+4

h) Find the inverse of the matrix $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}$ and use it to solve the system of equations $x - y + z = 1$, $x + y + 2z = 0$, $2x - y + 3z = 2$. 3+2

Group–B

2. Answer **any one** question: 3x1 = 3

a) Define Cartesian product of two sets. If $A = \{a,b,c,d\}$ and $B = \{a,b,c\}$ be two sets, determine $(A\times B)\cap(B\times A)$ and $(A\times B)\setminus(B\times A)$. 3

b) In a group $(G,\circ)$ if $a^2 = e$ for all $a \in G$, prove that G is commutative (e is the identity element of G). 3

Answer **any two** questions from Q. No. 3 to Q. No. 5: 5x2 = 10

3. a) Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be two mapping's defined as follows: $f(x)=x^2, x\in\mathbb{R};\ g(x)=x-2,\ x\in\mathbb{R}$, then show that $f\circ g\neq g\circ f$.
   b) In a group $(G,*)$, prove that $a*b=a*c$ implies $b=c$ for all $a,b,c\in G$. 3+2

4. Define subfield of a field. Is the ring $\left\{\begin{pmatrix} a & b \\ 2b & a\end{pmatrix}; a,b\in Q\right\}$ a field under matrix addition and matrix multiplication? Justify your answer. 1+4

5. State Cayley–Hamilton's theorem. If $A=\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0\end{pmatrix}$, verify that A satisfies its own characteristic equation. 1+4

Answer **any two** questions from Q. No. 6 to Q. No. 8: 6x2 = 12

6. Define subspace of a vectorspace. Show that the set $S=\{(x,y,z): x+y-z=0, 2x-y+z=0\}$ is a subspace of $\mathbb{R}^3$. Find the dimension of this subspace. 1+3+2

7. Find all the eigen values and any one of the eigen vectors of the matrix
$$\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -1 & -4 & -1\end{pmatrix}.$$
3+3

8. a) Prove that the set of all even integers forms a commutative ring with respective to usual addition and multiplication. 4
   b) Find k so that the vectors $(1,-1,2),(0,k,3)$ & $(-1,2,3)$ are linearly dependent in $\mathbb{R}^3$. 2

Group–C

Answer **any five** questions : 5×5 = 25

9. State the geometrical interpretation of $\dfrac{dy}{dx}$.

   A function f is defined by
$$\begin{aligned} f(x) &= x, && 0<x<1 \\ &= 2-x, && 1\le x\le 2 \\ &= x-\tfrac{1}{2}x^2, && x>2. \end{aligned}$$
   Show that $f'(2)$ exists but $f'(1)$ does not exist. 1+4

10. State Euler's theorem on homogeneous functions of two variables.

If $u = \cos^{-1}\left\{\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right\}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$. 1+4

11. Let $f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0 & , \ x^2+y^2 = 0 \end{cases}$

Examine whether $f_{xy}(0,0) = f_{yx}(0,0)$ or not. 5

12. If $y = (\sin^{-1}x)^2$ then prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$, where $y_n = \dfrac{d^n y}{dx^n}$. 5

13. If $u(x,y) = f(x^2+2yz,\ y^2+2zx)$ then prove that

$$\left(y^2 - zx\right)\frac{\partial u}{\partial x} + \left(x^2 - yz\right)\frac{\partial u}{\partial y} + \left(z^2 - xy\right)\frac{\partial u}{\partial z} = 0.$$ 5

14. Show that the radius of curvature of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at an extremity of the major axis is equal to half of its Latus recturm. 5

15. Show that at any point of the curve $x^{m+n} = k^{m-n}y^{2n}$, the *m*th power of subtangent varies as the *n*th power of the subnormal. 5

16. Show that the pedal equation of the lemniscate $r^2 = a^2\cos 2\theta$ si $r^3 = pa^2$. 5